

Fuzzy Finite Element Approach for Analysis of Fiber-Reinforced Laminated Composite Beams

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The modeling, analysis, and design of large and complex composite structures involve the use of the finite element approach. In most practical applications the structural and material parameters vary considerably and are subject to uncertainties, mainly because of the uncontrollable aspects associated with the manufacturing and assembly of composite materials. In many cases, the probabilistic methods cannot be applied because the probability distributions of the uncertain parameters are not usually known. Also, in some situations the parameters are known only in linguistic form. In this work, the uncertainties encountered in composite structures, which can be described in linguistic terms or in imprecise form, are modeled as fuzzy parameters. A fuzzy finite element analysis technique is developed for the analysis of laminated beams, involving fuzziness, possibly in the boundary conditions as well. The fuzzy beam element can undergo axial, bending, and transverse shear deformations. Using the basic concepts of the deterministic finite element theory, as well as fuzzy computations and fuzzy matrix operations, a fuzzy beam element is developed. The fuzzy beam element is used for the static and eigenvalue analysis of beams involving imprecise data and/or information. A numerical example is presented to demonstrate the feasibility and applicability of the methodology presented.

I. Introduction

FIBER-REINFORCED composite materials have emerged as a major class of structural materials in recent years and are either used or being considered as substitutes for metals in many weight-critical components in aerospace and other industries. For large and complex composite structures, the finite element analysis is used for modeling, analysis, and design. In fiber-reinforced composite structures, the parameters such as elastic modulus, fiber volume fraction, and geometry are not precisely known, and the available information might be vague, imprecise, qualitative, linguistic, or incomplete. This is mainly caused by the complex processes and the human judgment involved. During the manufacturing process, a number of factors, including material-related parameters such as resin chemistry and cure temperature and pressure, will affect the final characteristics of the composite material. Some of these factors such as the degree of cure and process time can only be experimentally determined and hence are not precisely known. Similarly the desired fiber orientation angles in the plies, and the geometric dimensions, for example, cannot be achieved accurately because of the machine tolerances involved. Thus most of the parameters of a composite structure are imprecise and involve uncertainties. Information such as, "Young's modulus of the material is in the range 3.5 GPa to 3.6 GPa," "fiber volume fraction is about 5%," and "the beam length is slightly larger than 2 m," cannot be handled conveniently by deterministic and probabilistic approaches. If approximate values of the parameters are used in finding the response parameters of the composite structure, the accuracy and reliability of the results cannot be ensured. The probabilistic approach requires considerable amount of information to find the probability distributions of the uncertain parameters, which in many cases can prove to be impossible or unrealistic. The fuzzy set theory can be conveniently used to model the uncertainties of composite structures. Establishing a rational fuzzy finite element analysis of laminated composite beams will be very useful for the analysis of composite structures.

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A. Stochastic Finite Element Method

Although there is no suitable technique available for the analysis of all types of imprecision, the stochastic finite element method can be used to handle uncertain parameters that are described by probability distributions. The stochastic finite element method was developed in the 1980s to account for uncertainties in the system parameters, geometry, and external actions. The uncertain variables were modeled as random variables/random fields with known characteristics. In 1980, a generic stochastic finite element method was proposed by Contreras¹ for modeling and analyzing structures in a probabilistic framework. The transient structural loads, idealized as stochastic processes, were incorporated into the finite element dynamic models with uncertain parameters. Nakagiri et al.² presented a method for the uncertain eigenvalue analysis of fiber-reinforced-plastic plates. By treating the fiber orientations and thicknesses of plies as random variables, the coefficients of variation of the eigenfrequency were evaluated. Vanmarcke and Grigoriu³ developed a method of stochastic finite element analysis for solving simple beam problems with random elastic modulus. Nakagiri and Hisada⁴ found the natural frequencies of beams with uncertain material properties or boundary conditions, whereas Liu et al.^{5,6} investigated static and dynamic responses of geometrically and materially nonlinear beam and truss structures with uncertain yield stresses. A common approach used in the stochastic finite element method is based on the mean-centered second-order perturbation technique. Such a method, equipped with the versatile nature of finite elements, has been shown to be a practical non-statistical approach among the probabilistic methods. A review of the developments of the various stochastic finite element methods can be found in the paper by Vanmarcke et al.⁷ and the books by Ghanem and Spanos⁸ and Haldar and Mahadevan,⁹ among others.

B. Fuzzy Finite Element Method

Akpan et al.¹⁰ presented a fuzzy finite element approach for modeling smart structures with vague or imprecise parameters in which fuzzy sets were used to represent the uncertainties present in the piezoelectric, mechanical, thermal, and physical properties of the smart structure. In Ref. 11, Akpan et al. presented a practical approach for analyzing the response of structures with fuzzy parameters. The methodology integrated the finite element modeling, response surface analysis and fuzzy analysis. Rao and Sawyer¹² presented a fuzzy finite element approach for the analysis

of imprecisely defined systems. Simple stress analysis problems involving vaguely defined geometry, material properties, loads and boundary conditions were solved to establish and illustrate the procedure. The approach developed is applicable to systems described in linguistic terms as well as those described by incomplete information. Chen and Rao¹³ presented a methodology using the fuzzy finite element method (FEM) for the vibration analysis of imprecisely defined systems. Numerical procedures to implement the methodology were presented. The longitudinal vibration of a three-stepped bar and the vibration of a 25-bar space truss were considered to illustrate the computational effectiveness of the approach. Muhanna and Mullen¹⁴ presented a fuzzy finite element formulation for the treatment of uncertainties in continuum mechanics. The approach was applied to treat the uncertainties present in the system in terms of load, geometry, and material properties in a number of examples. Rao and Weintraub¹⁵ considered several analytical problems where the information available was incomplete, uncertain, or involved user preferences. A methodology for fuzzy finite element analysis was described, and comparisons to the stochastic procedure were made where applicable. Results for bars, beams, plates, and thermal problems were discussed. Although some researchers have addressed the topic of fuzzy finite element method, this work represents the first application of the method to composite structures.

The fuzzy finite element method follows the same steps as the deterministic one,^{16,17} except that the element domain is a fuzzy domain and the mathematical operations used are based on fuzzy computations. This involves the evaluation of integrals of fuzzy functions over fuzzy domain that leads to a system of fuzzy linear equations. The resulting fuzzy linear equations, in the case of static analysis, are solved using a modified Gauss–Jordan elimination technique. A fuzzy Jacobi method is developed for the solution of the fuzzy eigenvalue problem. The finite element mesh is based on the crisp values of the geometry parameters, whereas the development of the element and system matrices involves fuzzy material, geometry, and load parameters. The static displacement and eigenvalue problems are solved by formulating the fuzzy equations $[K]X = P$ and $\omega^2[M]X = [K]X$, where X is the nodal displacement vector $[K]$ is the stiffness matrix, $[M]$ is the mass matrix, and ω is the natural frequency of the beam.

II. Basic Concepts of Fuzzy Numbers and Fuzzy Set Theory

Let X be a classical (crisp) set of objects, called the universe, whose generic elements are denoted x . Membership in a classical subset A of X can be viewed as a characteristic function μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad (1)$$

The set $\{0, 1\}$ is called a valuation set. A set A is called a fuzzy set if the valuation set is allowed to be the real interval $[0, 1]$. The fuzzy set A is completely characterized by the set of pairs

$$A = \{[x, \mu_A(x)], x \in X\} \quad (2)$$

where $\mu_A(x)$ is called the grade of membership function or degree of compatibility of x in A . The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A . For example, let $X = \{60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80\}$ be possible temperature settings of the thermostat ($^{\circ}\text{F}$) in an air-conditioned building. Then the fuzzy set A of “comfortable temperatures for human activity” can be defined by certain individual as

$$A = \{(60, 0), (62, 0.4), (64, 0.7), (66, 0.8), (68, 0.95), (70, 0.75), (72, 0.65), (74, 0.5), (76, 0.4), (78, 0.2), (80, 0.1)\} \quad (3)$$

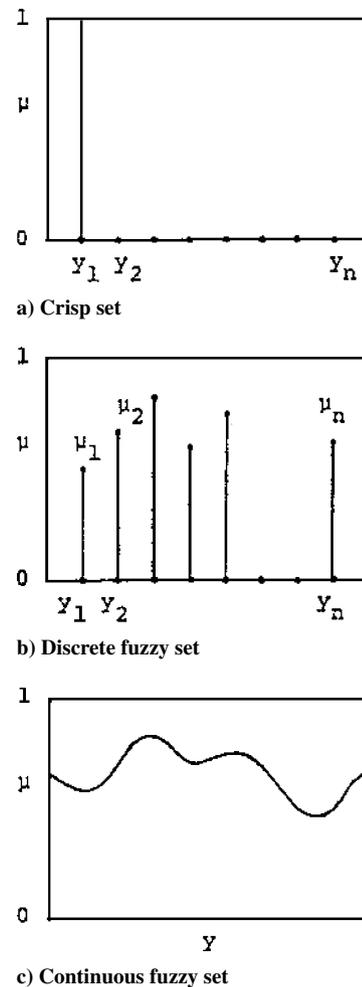


Fig. 1 Crisp and fuzzy sets.

where a grade of membership of 1 implies complete comfort and 0 implies complete discomfort. Clearly, A is a subset of X that has no sharp boundary. When X is a finite set $\{x_1, \dots, x_n\}$, a fuzzy set on X can also be expressed as

$$A = \mu_A(x_1)|_{x_1} + \mu_A(x_2)|_{x_2} + \dots + \mu_A(x_n)|_{x_n} = \sum_{i=1}^n \mu_A(x_i)|_{x_i} \quad (4)$$

When X is continuous, A can be expressed as

$$A = \int_x \mu_A(x)|_x \quad (5)$$

Crisp set theory is concerned with membership or nonmembership of precisely defined sets and is suitable for describing objective matters with countable events. The crisp set theory is accomplished by binary statements and is illustrated in Fig. 1a, which shows that there is only support for y_1 , without any ambiguity. Because the fuzzy set theory is concerned with linguistic statements of support for membership in imprecise sets, a discrete fuzzy set is denoted as in Fig. 1b, where the support for the membership $\mu_1, \mu_2, \dots, \mu_n$ is shown against their respective discrete values y_1, y_2, \dots, y_n . The discrete fuzzy set can be generalized to a continuous form as indicated in Fig. 1c.

The basic crisp set operations of union, intersection, and complement can be represented on Venn diagrams. Similar operations are necessary for dealing with fuzzy sets. However, because the sets A and B do not have the clarity of crisp A and B , they required a shading at the set boundaries in the Venn diagram depending on the linguistic statements made. Hence, for convenience, the graphs

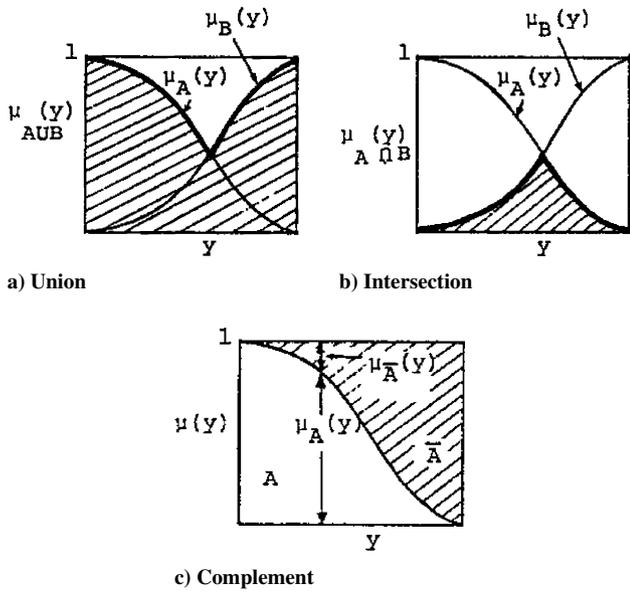


Fig. 2 Fuzzy set operations.

of μ_A and μ_B are used as a representation in order to visualize the set-theoretic operators of fuzzy sets.

When two fuzzy sets A and B with the corresponding supports for membership $\mu_A(y)$,

$$\begin{aligned} \mu_{A \cup B}(y) &= \mu_A(y) \vee \mu_B(y) = \max[\mu_A(y), \mu_B(y)] \\ &= \begin{cases} \mu_A(y), & \text{if } \mu_A \triangleright \mu_B \\ \mu_B(y), & \text{if } \mu_A \triangleleft \mu_B \end{cases} \end{aligned} \quad (6)$$

The result of this operation is shown in Fig. 2a. The fuzzy intersection is defined as

$$\begin{aligned} \mu_{A \cap B}(y) &= \mu_A(y) \wedge \mu_B(y) = \min(\mu_A(y), \mu_B(y)) \\ &= \begin{cases} \mu_A(y), & \text{if } \mu_A \triangleleft \mu_B \\ \mu_B(y), & \text{if } \mu_A \triangleright \mu_B \end{cases} \end{aligned} \quad (7)$$

The operation is shown in Fig. 2b. The complement of a fuzzy set A is shown as \bar{A} in Fig. 2c, in which to every $\mu_A(y)$ there corresponds $\mu_{\bar{A}}(y) = 1 - \mu_A(y)$, which defines the complement of the subset A, \bar{A} .

α Cuts: When an element $x \in X$ that typically belongs to a fuzzy set A is to be exhibited, it might be required to have its membership value greater than some threshold $\alpha \in [0, 1]$. The original set of such elements is called the α cut A_α of A .

Fuzzy arithmetic operations include fuzzy addition, subtraction, multiplication, and division. In this work, fuzzy arithmetic operations are denoted as (**), where ** represents deterministic arithmetic operations such as +, -, *, /. Thus, + denotes the deterministic addition, and (+) represents the fuzzy addition; similarly, x indicates a deterministic number, and (x) denotes a fuzzy number. Fuzzy arithmetic operations have features that are different from those of deterministic arithmetic. Fuzzy addition and fuzzy multiplication are commutative, associative, and distributive, but neither fuzzy subtraction nor fuzzy division is associative because $A(-)B(+)B \neq A$, and $[A(/)B](\cdot)B \neq A$. Also, a fuzzy zero (0) [or fuzzy one (1)] is defined as a fuzzy number in which the value zero has a membership value of one; the left and right numbers of zero (one) might not be the same. The fuzzy arithmetic operation of two fuzzy numbers A and B is defined as^{18,19}

$$A(**)B = \mu_{A(**)B}(z) = \bigvee_{z=x^{**}y} [\mu_A(x) \wedge \mu_B(y)] \quad (8)$$

which can also be expressed as

$$A_\alpha(+)B_\alpha = [a_1^\alpha, a_2^\alpha](+)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$$

$$A_\alpha(-)B_\alpha = [a_1^\alpha, a_2^\alpha](-)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha]$$

$$A_\alpha(\cdot)B_\alpha = [a_1^\alpha, a_2^\alpha](\cdot)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha \cdot b_1^\alpha, a_2^\alpha \cdot b_2^\alpha]$$

$$A_\alpha(/)B_\alpha = [a_1^\alpha, a_2^\alpha](/)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha / b_2^\alpha, a_2^\alpha / b_1^\alpha]$$

$$0 \notin [b_1^\alpha, b_2^\alpha] \quad (9)$$

where A_α and B_α are the intervals of confidence of A and B , respectively, for the level of presumption $\alpha, \alpha \in [0, 1]$, and $A_\alpha = \{x | \mu_A(x) \geq \alpha\}$ and $B_\alpha = \{x | \mu_B(x) \geq \alpha\}$. The trigonometry operations, involving $X_\alpha = [x_1^\alpha, x_2^\alpha]$, are defined in the first quadrant as

$$\cos(X_\alpha) = [\cos(x_2^\alpha), \cos(x_1^\alpha)], \quad x_1^\alpha, x_2^\alpha \in [0, \pi/2]$$

$$\sin(X_\alpha) = [\sin(x_1^\alpha), \sin(x_2^\alpha)], \quad x_1^\alpha, x_2^\alpha \in [0, \pi/2] \quad (10)$$

For other quadrants, suitable expressions can be developed similarly. If one of the operands is a deterministic number $k \in \mathbf{R}^+$, and the other one is a fuzzy number $X_\alpha = [x_1^\alpha, x_2^\alpha]$, then the multiplication result will be

$$k(\cdot)X = [k, k](\cdot)[x_1^\alpha, x_2^\alpha] = [k \cdot x_1^\alpha, k \cdot x_2^\alpha] \quad (11)$$

Fuzzy sets can also be used to manipulate linguistic variables. The linguistic variables—including labels such as small, big, low, and high; hedges such as very, quite, and extremely; negation (not); and connectives (and, but, or)—can be assembled into relatively complex statements such as very low or not very high, and their fuzzy representations can be compounded with the operations indicated earlier. Fuzzy hedge operations differ from arithmetic operations because they do not affect the values contained within a fuzzy number. A hedge operates only on the membership function {power of the membership function of A : $\mu_{\text{hedge}}(x) = [\mu_A(x)]^y$, y is a positive real number} of the fuzzy number A . When y is less than one, it is called concentration; and when y is greater than one, it is called dilation. For example, low and high are two fuzzy sets; the fuzzy representation of typical linguistic statements associated with low and high are shown in Table 1.

Fuzzy integration can be one of three types: The first one involves a fuzzy function integrated over a crisp range $[a, b]$; the second one involves a crisp function integrated over a fuzzy range $[A, B]$; and the last one involves a fuzzy function integrated over a fuzzy range $[A, B]$. In the first type, the fuzzy function $f(X)$ can always be treated as a fuzzy number with its membership function as $\mu_{f(X)}(y) = \alpha$ with $\alpha \in [0, 1]$ when X is a fuzzy variable. For each α , known as α cut, there corresponds an interval of y with $y \in [f_\alpha^-(x), f_\alpha^+(x)]$, where f_α^- and f_α^+ denote the lower and upper bounds of the interval, respectively. Figure 3 shows a fuzzy function (only a few points are shown). When $\alpha = 1$, $f(X)$ has the crisp feature.

Table 1 Fuzzy representation of typical linguistic statements

Set	Linguistic statement	Fiber content x					
		1	2	5	7	9	11
A	Low	1.0	1.0	0.8	0.5	0.3	0.0
\bar{A}	Not low	0.0	0.0	0.2	0.5	0.7	1.0
A^2	Very low	1.0	1.0	0.64	0.25	0.09	0.0
A^4	Very very low	1.0	1.0	0.4096	0.0625	0.0081	0.0
\bar{A}^2	Not very low	0.0	0.0	0.36	0.75	0.91	1.0
B	High	0.0	0.2	0.4	0.7	1.0	1.0
B^2	Very high	0.0	0.04	0.16	0.49	1.0	1.0
$A^2 \cup B^2$	Very low or very high	1.0	1.0	0.64	0.49	1.0	1.0
$A \cap \bar{A}^2$	Low but not very low	0.0	0.0	0.36	0.5	0.3	0.0

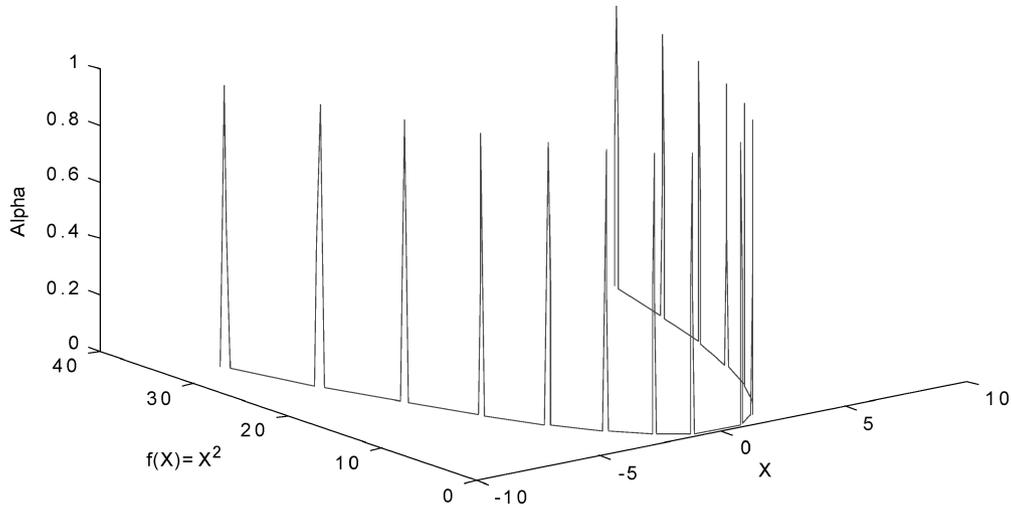


Fig. 3 Graphical representation of a fuzzy function $f(X)$.

Thus the integral is given by

$$I(a, b)_\alpha = \left[\int_a^b f_\alpha^-(x) dx, \int_a^b f_\alpha^+(x) dx \right] \quad (12)$$

In the second type, the following function is evaluated:

$$C(z) = \int_A^B f(\xi) d\xi = \bigvee_{z = \int_x^y f(\xi) d\xi} [\mu_A(x) \wedge \mu_B(y)] \quad (13)$$

where C is a fuzzy number on the universe of discourse z , and $z = x \times y$ is the product space of x and y . In the third type, when a fuzzy function is integrated within a fuzzy range the following function is evaluated:

$$C(z)_\alpha = \int_A^B f(X) dX = \left\{ \bigvee_{z = \int_x^y f_\alpha^-(\xi) d\xi} [\mu_A(x) \wedge \mu_B(y)], \right. \\ \left. \times \bigvee_{z = \int_x^y f_\alpha^+(\xi) d\xi} [\mu_A(x) \wedge \mu_B(y)] \right\} \quad (14)$$

It can be seen that for a fuzzy function $f(X)$ lower and upper bounds exist as $[f_\alpha^-, f_\alpha^+]$ for each α cut. The functions f_α^- and f_α^+ are integrated over the fuzzy range $[A, B]$ as in Eq. (13), to evaluate the integral.

Differentiation of a fuzzy function $f(X)$ is considered as follows. When $f(X)$ is a nonnegative, continuous fuzzy function that increases monotonically in the segment $[a, b] \subset R^+$, the derivative will be nonnegative in this segment. For $\forall x \in [x_\alpha^-, x_\alpha^+] \subset [a, b] \subset R^+$, $f(X)_\alpha = f([x_\alpha^-, x_\alpha^+]) = [f(x_\alpha^-), f(x_\alpha^+)]$, $\alpha \in [0, 1]$, the derivative of $f(X)$ is defined as

$$f'(X)_\alpha = [f'(x_\alpha^-), f'(x_\alpha^+)] \quad (15)$$

It can be seen that both $f(X)$ and $f'(X)$ are convex. From the extension principle, the deterministic rules of differentiation of a function $f(x)$ over x can be used except that all of the variables now are fuzzy numbers so long as the function f is a nonnegative, continuous function of x that increases monotonically within some range. If x , f , or $f'(x)$ change signs at some point, then a new subrange is defined within the original range to make sure that within the new subrange the function is monotonic. Thus the final result will be a union of several different results each corresponding to a subrange.

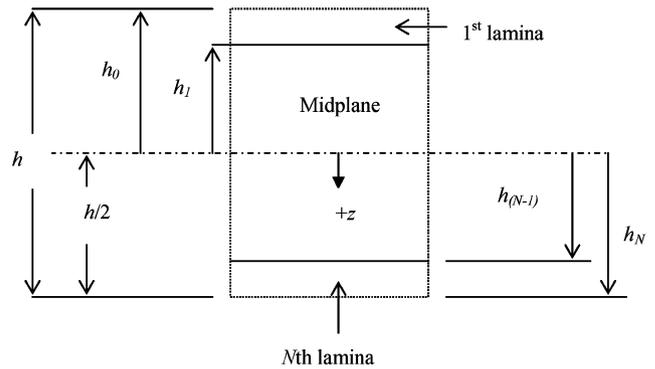


Fig. 4 Laminate geometry.

A fuzzy matrix is a matrix whose elements are fuzzy numbers. A system of fuzzy linear equations with the coefficient matrix A and the right-hand side vector b fuzzy is expressed as $A(\cdot)X = b$. The necessary and sufficient conditions for the existence of a unique solution to a single fuzzy algebraic equation are given in Ref. 12. These conditions state that in the fuzzy expression $A(\cdot)X = b$ the relative spread of b , defined as $(b^+/b^-)_\alpha$ for every level α , must be greater than or equal to the relative spread of A for a solution X to exist, where b^+ and b^- denote the upper and lower bounds for every level α , respectively. In fuzzy matrix operation, if the condition $[A(/)B](\cdot)(B)$ is met it will not be equal to A . Hence, a modification is made as follows. If $C = Z(-)[A(/)B](\cdot)(B)$, it is modified as $C = [1/(\text{crisp value of } B)](\cdot) ([Z(\cdot)B](\cdot)[A(\cdot)B])$. Because this modification denotes an elementary operation, it will not affect the final result. Because a fuzzy variable is a fuzzy set, it is possible to locate it inside a closed interval of confidence of R : $X_\alpha = [a_1^\alpha, a_2^\alpha]$ for each α level, and $\alpha \in [0, 1]$, with $a_1^\alpha \leq a_2^\alpha$ (Ref. 1). Each parameter of a composite structure can be treated as a fuzzy variable. Therefore, all of the computations can be performed in fuzzy form, and fuzzy computational results related to composite structure also fall within certain ranges.

III. Fuzzy Laminated Beam Theory

A laminate is constructed by stacking a number of laminae in the thickness z direction. The geometric midplane of the laminate is considered to be in the xy plane with the z axis defining the thickness direction. The total thickness of the laminate is h , and the total number of laminae is N as shown in Fig. 4. Assuming that all of the laminae are macroscopically homogeneous and behave in a linearly elastic manner, the laminate strains are linearly related to

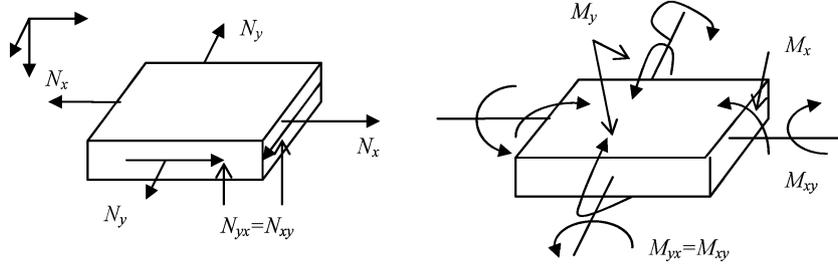


Fig. 5 In-plane bending and twisting loads applied on a laminate.

the distance from the midplane as

$$\begin{Bmatrix} \varepsilon_{xx\alpha}^\circ \\ \varepsilon_{yy\alpha}^\circ \\ \gamma_{xy\alpha}^\circ \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x_\alpha} (+) \frac{1}{2} (\cdot) \left(\frac{\partial w_0}{\partial x} \right)^2 \alpha \\ \frac{\partial v_0}{\partial y_\alpha} (+) \frac{1}{2} (\cdot) \left(\frac{\partial w_0}{\partial y} \right)^2 \alpha \\ \frac{\partial u_0}{\partial x_\alpha} (+) \frac{\partial v_0}{\partial x_\alpha} (+) \frac{\partial w_0}{\partial x_\alpha} (\cdot) \frac{\partial w_0}{\partial y_\alpha} \end{Bmatrix}$$

$$\begin{Bmatrix} k_{xx\alpha} \\ k_{yy\alpha} \\ k_{xy\alpha} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x_\alpha^2} \\ -\frac{\partial^2 w_0}{\partial y_\alpha^2} \\ -2(\cdot)z(\cdot) \frac{\partial^2 w_0}{\partial x \partial y_\alpha} \end{Bmatrix}$$

$$\begin{aligned} \varepsilon_{xx\alpha} &= \varepsilon_{xx\alpha}^\circ (+)z(\cdot)k_{xx\alpha} \\ \varepsilon_{yy\alpha} &= \varepsilon_{yy\alpha}^\circ (+)z(\cdot)k_{yy\alpha} \\ \gamma_{xy\alpha} &= \gamma_{xy\alpha}^\circ (+)z(\cdot)k_{xy\alpha} \end{aligned} \quad (16)$$

where ε_{xx}° and ε_{yy}° are the midplane normal strains, γ_{xy}° is the midplane shear strain, k_{xx} and k_{yy} are the bending curvatures, k_{xy} is the twisting curvature, and z is the distance from the midplane in the thickness direction of the laminate. If δU , δV , and δK denote the variations of strain energy, work done by applied forces, and kinetic energy, respectively, and $(\delta u_0, \delta v_0, \delta w_0)$ represent the variations in displacements, the Hamilton's principle^{20,21} (which can be considered as the dynamic version of the principle of virtual displacements) and the fuzzy form of Euler–Lagrange equations of the classical plate theory can be expressed as in Eq. (17). Let each layer be orthotropic with respect to its material symmetry line so that the relationship between the applied force and moment resultants (shown in Fig. 5) and the midplane strains and curvatures of the laminate are given by Eqs. (18) and (19).

$$\int_0^T [\delta U (+) \delta V (-) \delta K] dt = (0)$$

$$\delta u_0 : \frac{\partial N_x}{\partial x} (+) \frac{\partial N_{xy}}{\partial y} = I_0(\cdot) \frac{\partial^2 u_0}{\partial t^2} (-) I_1(\cdot) \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right)$$

$$\delta v_0 : \frac{\partial N_{xy}}{\partial x} (+) \frac{\partial N_y}{\partial y} = I_0(\cdot) \frac{\partial^2 v_0}{\partial t^2} (-) I_1(\cdot) \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial y} \right)$$

$$\delta w_0 : \frac{\partial^2 M_x}{\partial x^2} (+) 2(\cdot) \frac{\partial^2 M_{xy}}{\partial y \partial x} (+) \frac{\partial^2 M_y}{\partial y^2} (+) \Omega(w_0) (+) q = I_0(\cdot) \frac{\partial^2 w_0}{\partial t^2}$$

$$(-) I_2(\cdot) \frac{\partial^2}{\partial t^2} \left[\frac{\partial^2 w_0}{\partial x^2} (+) \frac{\partial^2 w_0}{\partial y^2} \right] (+) I_1(\cdot) \frac{\partial^2}{\partial t^2} \left[\frac{\partial u_0}{\partial x} (+) \frac{\partial v_0}{\partial y} \right]$$

$$\begin{aligned} \Omega(w_0) &= \frac{\partial}{\partial x} \left[N_x(\cdot) \frac{\partial w_0}{\partial x} (+) N_{xy}(\cdot) \frac{\partial w_0}{\partial y} \right] \\ & (+) \frac{\partial}{\partial y} \left[N_{xy}(\cdot) \frac{\partial w_0}{\partial x} (+) N_y(\cdot) \frac{\partial w_0}{\partial y} \right] \end{aligned} \quad (17)$$

$$\begin{bmatrix} N_{x\alpha} \\ N_{y\alpha} \\ N_{xy\alpha} \end{bmatrix} = [A] \begin{bmatrix} \varepsilon_{xx\alpha}^\circ \\ \varepsilon_{yy\alpha}^\circ \\ \varepsilon_{xy\alpha}^\circ \end{bmatrix} (+) [B] \begin{bmatrix} k_{xx\alpha} \\ k_{yy\alpha} \\ k_{xy\alpha} \end{bmatrix} \quad (18)$$

$$\begin{bmatrix} M_{x\alpha} \\ M_{y\alpha} \\ M_{xy\alpha} \end{bmatrix} = [B] \begin{bmatrix} \varepsilon_{xx\alpha}^\circ \\ \varepsilon_{yy\alpha}^\circ \\ \varepsilon_{xy\alpha}^\circ \end{bmatrix} (+) [D] \begin{bmatrix} k_{xx\alpha} \\ k_{yy\alpha} \\ k_{xy\alpha} \end{bmatrix} \quad (19)$$

where

$$A_{mn\alpha} = \sum_{j=1}^N (\bar{Q}_{mn\alpha})_j (\cdot) [h_{j\alpha} (-) h_{j-1\alpha}] \quad (20)$$

$$B_{mn\alpha} = \left(\frac{1}{2} \right) (\cdot) \sum_{j=1}^N (\bar{Q}_{mn\alpha})_j (\cdot) [(h_{j\alpha})^2 (-) (h_{j-1\alpha})^2] \quad (21)$$

$$D_{mn\alpha} = \left(\frac{1}{3} \right) (\cdot) \sum_{j=1}^N (\bar{Q}_{mn\alpha})_j (\cdot) [(h_{j\alpha})^3 (-) (h_{j-1\alpha})^3] \quad (22)$$

where N_x is the normal force resultant in the x direction (per unit width), N_y is the normal force resultant in the y direction (per unit width), N_{xy} is the shear force resultant (per unit width), M_x is the bending moment resultant in the yz plane (per unit width), M_y is the bending moment resultant in the xz plane (per unit width), M_{xy} is the twisting moment resultant (per unit width), $[A]$ is the extensional stiffness matrix for the laminate (N/m or lb/in.), $[B]$ is the coupling stiffness matrix for the laminate (N or lb), and $[D]$ is the bending stiffness matrix for the laminate (N-m or lb-in.). N is the total number of laminas in the laminate, j denotes the j th lamina, $(\bar{Q}_{mn})_j$ is the element in the $[\bar{Q}]$ matrix of the j th lamina, h_j is the distance from the midplane to the top of the j th lamina, and h_{j-1} is the distance from the midplane to the bottom of the j th lamina.

If the normal force and moment resultants acting on a laminate are known, its midplane strains and curvatures can be given by inverting Eqs. (18) and (19) (Refs. 21 and 22):

$$\begin{bmatrix} \varepsilon_{xx\alpha}^\circ \\ \varepsilon_{yy\alpha}^\circ \\ \gamma_{xy\alpha}^\circ \end{bmatrix} = [A_{1\alpha}] \begin{bmatrix} N_{xx\alpha} \\ N_{yy\alpha} \\ N_{xy\alpha} \end{bmatrix} (+) [B_{1\alpha}] \begin{bmatrix} M_{xx\alpha} \\ M_{yy\alpha} \\ m_{xy\alpha} \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} k_{xx\alpha} \\ k_{yy\alpha} \\ k_{xy\alpha} \end{bmatrix} = [C_{1\alpha}] \begin{bmatrix} N_{xx\alpha} \\ N_{yy\alpha} \\ N_{xy\alpha} \end{bmatrix} (+) [D_{1\alpha}] \begin{bmatrix} M_{xx\alpha} \\ M_{yy\alpha} \\ m_{xy\alpha} \end{bmatrix} \quad (24)$$

$$\begin{aligned}
 [A_{1\alpha}] &= [A_\alpha^{-1}](+)[A_\alpha^{-1}](\cdot)[B_\alpha](\cdot)[(D_\alpha^*)^{-1}](\cdot)[B_\alpha](\cdot)[A_\alpha^{-1}] \\
 [B_{1\alpha}] &= [(0)](-)[A_\alpha^{-1}](\cdot)[B_\alpha](\cdot)[(D_\alpha^*)^{-1}] \\
 [C_{1\alpha}] &= [(0)](-)[(D_\alpha^*)^{-1}](\cdot)[B_\alpha](\cdot)[A_\alpha^{-1}] = [B_{1\alpha}]^T \\
 [D_\alpha^*] &= [D_\alpha](-)[B_\alpha](\cdot)[A_\alpha^{-1}](\cdot)[B_\alpha] \\
 [D_{1\alpha}] &= [(D_\alpha^*)^{-1}]
 \end{aligned} \tag{25}$$

Note that for a symmetric laminate, $[B] = [(0)]$. Knowing the midplane strains and curvatures for the laminate, the strains at the midplane of each lamina can be calculated from relationship

$$\begin{bmatrix} \varepsilon_{xx\alpha} \\ \varepsilon_{yy\alpha} \\ \gamma_{xy\alpha} \end{bmatrix}_j = \begin{bmatrix} \varepsilon_{xx\alpha}^\circ \\ \varepsilon_{yy\alpha}^\circ \\ \gamma_{xy\alpha}^\circ \end{bmatrix} (+) z_{j\alpha} (\cdot) \begin{bmatrix} k_{xx\alpha} \\ k_{yy\alpha} \\ k_{xy\alpha} \end{bmatrix} \tag{26}$$

where z_j is the fuzzy distance from the laminate midplane to the midplane of the j th lamina.

IV. Fuzzy Finite Element Analysis

From the classical laminate plate theory, the displacements (u, v, w) are defined by

$$\begin{aligned}
 u(x, y, z, t) &= u_0(x, y, t)(-)z(\cdot)\frac{\partial w_0}{\partial x} \\
 v(x, y, z, t) &= v_0(x, y, t)(-)z(\cdot)\frac{\partial w_0}{\partial y} \\
 w(x, y, z, t) &= w_0(x, y, t)
 \end{aligned} \tag{27}$$

where (u_0, v_0, w_0) are the displacements of the laminate midplane along x, y , and z directions, respectively. Equation (27) shows that the form of the displacement allows reduction of the three-dimensional problem to one of studying the deformation of the reference plane $z = (0)$ (or the midplane). Once the midplane displacements (u_0, v_0, w_0) are known, the displacements of any arbitrary point (x, y, z) in the three-dimensional continuum can be determined using Eq. (27). To analyze the composite laminated beam elastostatic response, a line element with two degrees of freedom at each node is employed (Fig. 6) with h denoting the length of the element and the numbers 1 and 2 indicating the node numbers. For the laminated beam analysis, when the beam is long enough the effect of the Poisson ratio and shear coupling on the deflection can be assumed to be negligible. The transverse deflection w_0 can be treated as a function of the coordinate x and time t only as

$$\begin{aligned}
 w_0[(0), t] &= W1(t), & \theta[(0)] &= -\frac{\partial w_0}{\partial x} \Big|_{x=(0)} = W2(t) \\
 w_0[(h), t] &= W3(t), & \theta[(h)] &= -\frac{\partial w_0}{\partial x} \Big|_{x=(h)} = W4(t)
 \end{aligned} \tag{28}$$

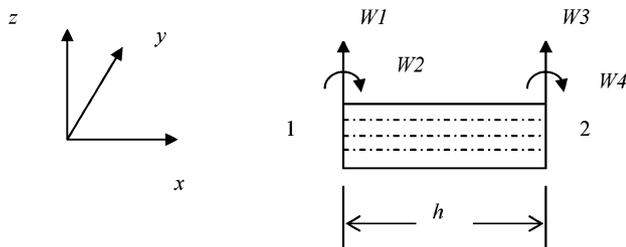


Fig. 6 One element of the beam.

The interpolation model within the element is given by

$$\begin{aligned}
 w_0[(0), t] &\approx W(x, t) = c_1(+)+c_2(\cdot)x(+)+c_3(\cdot)x^2(+)+c_4(\cdot)x^3 \\
 &= \sum_{j=1}^n w_j(t)(\cdot)\varphi_j(t) \\
 w_1 &= W[(0), t], & w_2 &= \theta[(0), t], & w_3 &= W[(h), t] \\
 w_4 &= \theta[(h), t]
 \end{aligned} \tag{29}$$

where $n = 4$ is the number of degrees of freedom of the beam element and φ_j is the shape function. In the finite element analysis, w is treated as a function of x with $x \in [(0), h^{(e)}]$, where $h^{(e)}$ is the length of the element e . Thus w can be chosen as a nonnegative, continuous function of x that increases monotonically in the range $[(0), h^{(e)}]$, and the derivative of fuzzy number $w(x)$ will be $w'(x)_\alpha = [w'(x_\alpha^1), w'(x_\alpha^2)]$, with $\alpha \in [0, 1]$. When we consider a range in which $w(x)$ and $w'(x)$ have the same sign, the deterministic rules of differentiation of a function can be used except that all of the variables now are fuzzy. For the integrations of a fuzzy function $f(X)$ over a fuzzy range $[A, B]$, Eq. (14) can be used. If $M_y = M_{xy} = (0)$, q is the distributed transverse load per unit width; b is the width of the laminate beam; ρ is the mass density; $\hat{q} = b(\cdot)q$; $\hat{I}_i = b(\cdot)I_i$;

$$I_i = \int_{(-d)/2}^{(d)/2} \rho(\cdot)z^i dz$$

$i = 0, 1, 2$; I_0 and I_2 are the mass moments of inertias; and d is the thickness of the laminate, the equation of motion of the laminated beam can be expressed as

$$\frac{\partial^2 M_x}{\partial x^2} (+) \hat{N}_{xx}(\cdot) \frac{\partial^2 w_0}{\partial x^2} (+) \hat{q} = \hat{I}_2(\cdot) \frac{\partial^2 w_0}{\partial t^2} (-) \hat{I}_2(\cdot) \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \tag{30}$$

where \hat{N}_{xx} is the applied axial force. When the laminate is symmetric,

$$M_x = -E_{xx}^b(\cdot)I_{yy}(\cdot)\frac{\partial^2 w_0}{\partial x^2}, \quad E_{xx}^b = \frac{12}{d^3(\cdot)D_{11}^*}, \quad I_{yy} = \frac{b(\cdot)d^3}{12} \tag{31}$$

By using the Hamilton's principle (or the dynamic version of the principle of virtual displacements),^{20,21} we can obtain

$$\begin{aligned}
 (0) &= \int_{(0)}^{(h)} \left\{ M_x(\cdot) \frac{d^2 \varphi_i}{dx^2}(\cdot) \left[\sum_{j=1}^4 w_j(t)(\cdot) \frac{d^2 \varphi_j}{dx^2} \right] \right. \\
 & (+) b(\cdot) \hat{N}_{xx}(\cdot) \frac{d\varphi_i}{dx} \left[\sum_{j=1}^4 w_j(t) \frac{d\varphi_j}{dx} \right] \\
 & (+) \hat{I}_0(\cdot) \varphi_i(\cdot) \left[\sum_{j=1}^4 \varphi_j(t)(\cdot) \frac{d^2 w_j}{dt^2} \right] \\
 & (+) \hat{I}_2(\cdot) \frac{d\varphi_i}{dx}(\cdot) \left[\sum_{j=1}^4 \frac{d\varphi_j}{dx}(\cdot) \frac{d^2 w_j}{dt^2} \right] (-) \varphi_i(\cdot) \hat{q} \Big\} dx \\
 & (-) Q_1(\cdot) \varphi_i[(0)] (-) Q_3(\cdot) \varphi_i[(h)] (-) Q_2(\cdot) \left(-\frac{d\varphi_i}{dx} \right) \Big|_{x=(0)} \\
 & (-) Q_4(\cdot) \left(-\frac{d\varphi_i}{dx} \right) \Big|_{x=(h)} = \sum_{j=1}^4 \int_{(0)}^{(h)} \left\{ \left(M_x(\cdot) \frac{d^2 \varphi_i}{dx^2}(\cdot) \frac{d^2 \varphi_j}{dx^2} \right. \right. \\
 & \left. \left. (+) b(\cdot) \hat{N}_{xx}(\cdot) \frac{d\varphi_i}{dx}(\cdot) \frac{d\varphi_j}{dx} \right) (\cdot) w_j (+) \left[\hat{I}_0(\cdot) \varphi_i(\cdot) \varphi_j \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & (+)\hat{I}_2(\cdot)\left[\frac{d\varphi_i}{dx}(\cdot)\frac{d\varphi_j}{dx}(\cdot)\right](\cdot)\left[\frac{d^2w_j}{dt^2}(-)\varphi_i(\cdot)\hat{q}\right]dx(-)Q_1(\cdot)\varphi_i(0) \\
 & (-)Q_3(\cdot)\varphi_i(h)(-)\left[\frac{d\varphi_i}{dx}\right]_{x=0} \\
 & (-)Q_4(\cdot)\left[\frac{d\varphi_i}{dx}\right]_{x=h} = \sum_{j=1}^4 [K_{ij}(+)G_{ij}](\cdot)w_j \\
 & (+)M_{ij}(\cdot)\left[\frac{d^2w_j}{dt^2}\right](-)F_i \tag{32} \\
 Q_1(t) &= \left\{ \frac{\partial}{\partial x} \left[M_x(\cdot)\frac{\partial^2 w_0}{\partial x^2}(-)\hat{I}_2(\cdot)\frac{\partial^2 w_0}{\partial t^2} \right](-)\hat{N}_{xx}(\cdot)\frac{\partial w_0}{\partial x} \right\} \Big|_{x=0} \\
 Q_3(t) &= -\left\{ \frac{\partial}{\partial x} \left[M_x(\cdot)\frac{\partial^2 w_0}{\partial x^2}(-)\hat{I}_2(\cdot)\frac{\partial^2 w_0}{\partial t^2} \right](-)\hat{N}_{xx}(\cdot)\frac{\partial w_0}{\partial x} \right\} \Big|_{x=h} \\
 Q_2(t) &= \left[M_x(\cdot)\frac{\partial^2 w_0}{\partial x^2} \right] \Big|_{x=0} \\
 Q_4(t) &= -\left[M_x(\cdot)\frac{\partial^2 w_0}{\partial x^2} \right] \Big|_{x=h} \tag{33} \\
 K_{ij} &= \int_0^h M_{xx}(\cdot)\frac{d^2\varphi_i}{dx^2}(\cdot)\frac{d^2\varphi_j}{dx^2}dx \\
 G_{ij} &= \int_0^h b(\cdot)N_{xx}(\cdot)\frac{d\varphi_i}{dx}(\cdot)\frac{d\varphi_j}{dx}dx \\
 M_{ij} &= \int_0^h \left[\hat{I}_0(\cdot)\varphi_i(\cdot)\varphi_j(+)\hat{I}_2(\cdot)\frac{d\varphi_i}{dx}(\cdot)\frac{d\varphi_j}{dx} \right] dx \\
 F_i &= q_i(+)\left[Q_1(\cdot)\varphi_i(0) \right] (+) \left[Q_3(\cdot)\varphi_i(h) \right] (+) \left[Q_2(\cdot)\left(-\frac{d\varphi_i}{dx}\right) \right]_{x=0} \\
 & (+) \left[Q_4(\cdot)\left(-\frac{d\varphi_i}{dx}\right) \right]_{x=h} \\
 q_i &= \int_0^h \varphi_i(\cdot)\hat{q}(\cdot)dx \tag{34}
 \end{aligned}$$

Equation (32) can be written in matrix form as $[K](+)[G](\cdot)\{w\}(+)[M](\cdot)\{w\} = \{F\}$ (35)

where $[K]$ is the stiffness matrix, $[G]$ is the geometric stiffness matrix, $[M]$ is the mass matrix, and $\{F\}$ is the force vector. $F_i^e = q_i^e(+)\varphi_i^e$, which is composed of nodal forces q_i^e as a result of the distributed load \hat{q} and the nodal reactions Q_i^e with all of the matrices being symmetric. From the integrals of Eq. (34), all of

the coefficient matrices can be obtained. For a symmetric beam, if the geometry and material properties are elementwise constant, the matrices can be expressed as

$$[K^e] = [2(\cdot)E_{xx}^e(\cdot)I_{yy}^e(\cdot)(h_e^3)] \tag{36}$$

$$(\cdot) \begin{bmatrix} 6 & -3(\cdot)h_e & -6 & -3(\cdot)h_e \\ -3(\cdot)h_e & 2(\cdot)h_e^2 & 3(\cdot)h_e & h_e^2 \\ -6 & 3(\cdot)h_e & 6 & 3(\cdot)h_e \\ -3(\cdot)h_e & h_e^2 & 3(\cdot)h_e & 2(\cdot)h_e^2 \end{bmatrix} \tag{36}$$

$$[q^e] = [q_0^e(\cdot)(h_e)(/12)](\cdot) \begin{bmatrix} 6 \\ -1(\cdot)h_e \\ 6 \\ h_e \end{bmatrix} \tag{37}$$

$$[M^e] = [\hat{I}_0^e(\cdot)(h_e)(/420)] \tag{38}$$

$$(\cdot) \begin{bmatrix} 156 & -22(\cdot)h_e & 54 & 13(\cdot)h_e \\ -22(\cdot)h_e & 4(\cdot)h_e^2 & -13(\cdot)h_e & -3(\cdot)h_e^2 \\ 54 & -13(\cdot)h_e & 156 & 22(\cdot)h_e \\ 13(\cdot)h_e & -3(\cdot)h_e^2 & 22(\cdot)h_e & 4(\cdot)h_e^2 \end{bmatrix} \tag{38}$$

$$(+) [\hat{I}_2^e(\cdot)(30(\cdot)h_e)](\cdot)[H^e] \tag{38}$$

$$[G^e] = [b_e(\cdot)\hat{N}_{xx}^e(\cdot)(/30(\cdot)h_e)](\cdot)[H^e] \tag{39}$$

$$[H^e] = \begin{bmatrix} 36 & -3(\cdot)h_e & -36 & -3(\cdot)h_e \\ -3(\cdot)h_e & 4(\cdot)h_e^2 & 3(\cdot)h_e & -1(\cdot)h_e^2 \\ -36 & 3(\cdot)h_e & 36 & 3(\cdot)h_e \\ -3(\cdot)h_e & -1(\cdot)h_e^2 & 3(\cdot)h_e & 4(\cdot)h_e^2 \end{bmatrix} \tag{40}$$

For static bending under applied transverse load and axial load, Eq. (35) takes the form

$$([K^e](+)[G^e])(\cdot)\{w^e\} = \{F^e\} \tag{41}$$

For the natural vibration of a beam, Eq. (35) will be transformed as

$$([K^e](\cdot)\omega^2(\cdot)[M^e])(\cdot)\{W^e\} = \{\bar{F}^e\} \tag{42}$$

with $w^e = W^e(\cdot)e^{-it(\cdot)\omega}$, $F^e = \bar{F}^e(\cdot)e^{-it(\cdot)\omega}$, $i = \sqrt{-1}$. After evaluating the element matrices and assembling them to derive the system matrices, the nodal displacements of the laminated beam can be determined by fuzzy Gauss–Jordan elimination method. External loads and boundary conditions will be applied as per the deterministic finite element analysis procedure. Once the nodal displacements are known, the complete elastostatic response of the beam can be computed through fuzzy computations of composite material mechanics. The natural frequencies of the beam can also be computed by the fuzzy Jacobi method.

V. Application and Discussion

To illustrate the application of fuzzy finite element analysis of laminated beams, an example is considered with the uncertainties of all parameters assumed to lie in the range $[-1\%, +0.5\%]$ about their respective deterministic values and the membership functions of the fuzzy parameters assumed to be of triangular form. The fuzzy analysis of the cantilever beam shown in Fig. 7 is considered. Each

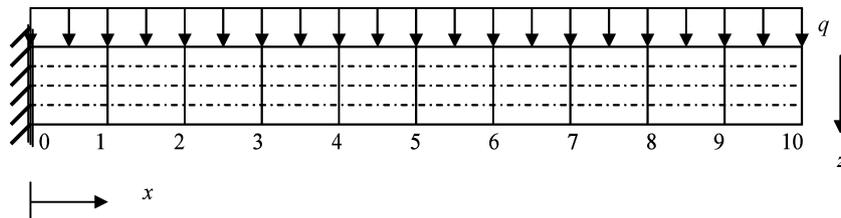


Fig. 7 Cantilever laminated beam.

Table 2 Stiffness matrix $[\bar{Q}]$ for $\theta = 45$ deg

α	\bar{Q}_{11} , Pa	\bar{Q}_{12} , Pa	\bar{Q}_{16} , Pa	\bar{Q}_{22} , Pa	\bar{Q}_{26} , Pa	\bar{Q}_{66} , Pa
0.0	3.95710e+010	3.29314e+010	3.06163e+010	3.93654e+010	3.05173e+010	3.38637e+010
0.2	3.96610e+010	3.30343e+010	3.07382e+010	3.94698e+010	3.06461e+010	3.39740e+010
0.4	3.99344e+010	3.33268e+010	3.10218e+010	3.98168e+010	3.09811e+010	3.42802e+010
0.6	4.00722e+010	3.35236e+010	3.12017e+010	4.00513e+010	3.12046e+010	3.44791e+010
0.8	4.01134e+010	3.35833e+010	3.12811e+010	4.01092e+010	3.12866e+010	3.45509e+010
1.0	4.01243e+010	3.35999e+010	3.13032e+010	4.01243e+010	3.13032e+010	3.45680e+010
0.8	4.01435e+010	3.36109e+010	3.13302e+010	4.01409e+010	3.13273e+010	3.45884e+010
0.6	4.02187e+010	3.36628e+010	3.14267e+010	4.01968e+010	3.14120e+010	3.46795e+010
0.4	4.04897e+010	3.38706e+010	3.16532e+010	4.03575e+010	3.16053e+010	3.48855e+010
0.2	4.08552e+010	3.41907e+010	3.19916e+010	4.06211e+010	3.18762e+010	3.51805e+010
0.0	4.09730e+010	3.42949e+010	3.21204e+010	4.07244e+010	3.19875e+010	3.52912e+010

Table 3 Stiffness matrix $[\bar{Q}]$ for $\theta = -45$ deg

α	\bar{Q}_{11} , Pa	\bar{Q}_{12} , Pa	\bar{Q}_{16} , Pa	\bar{Q}_{22} , Pa	\bar{Q}_{26} , Pa	\bar{Q}_{66} , Pa
0.0	3.95710e+010	3.29314e+010	-3.21204e+010	3.93654e+010	-3.19875e+010	3.38637e+010
0.2	3.96610e+010	3.30343e+010	-3.19916e+010	3.94698e+010	-3.18762e+010	3.39740e+010
0.4	3.99344e+010	3.33268e+010	-3.16532e+010	3.98168e+010	-3.16053e+010	3.42802e+010
0.6	4.00722e+010	3.35236e+010	-3.14267e+010	4.00513e+010	-3.14120e+010	3.44791e+010
0.8	4.01134e+010	3.35833e+010	-3.13302e+010	4.01092e+010	-3.13273e+010	3.45509e+010
1.0	4.01243e+010	3.35999e+010	-3.13032e+010	4.01243e+010	-3.13032e+010	3.45680e+010
0.8	4.01435e+010	3.36109e+010	-3.12811e+010	4.01409e+010	-3.12866e+010	3.45884e+010
0.6	4.02187e+010	3.36628e+010	-3.12017e+010	4.01968e+010	-3.12046e+010	3.46795e+010
0.4	4.04897e+010	3.38706e+010	-3.10218e+010	4.03575e+010	-3.09811e+010	3.48855e+010
0.2	4.08552e+010	3.41907e+010	-3.07382e+010	4.06211e+010	-3.06461e+010	3.51805e+010
0.0	4.09730e+010	3.42949e+010	-3.06163e+010	4.07244e+010	-3.05173e+010	3.52912e+010

Table 4 Results of example: $[+45/-45]_s$ angle-ply laminate with $[B] = \{0\}$

α	A_{11} , N/m	A_{12} , N/m	A_{22} , N/m	A_{66} , N/m	D_{11} , N · m	D_{12} , N · m	D_{16} , N · m	D_{22} , N · m	D_{26} , N · m	D_{66} , N · m
0.0	9.55577e+008	7.97081e+008	9.52562e+008	8.19739e+008	45,750.1	38,118.5	26,435.3	45,541.9	26,357.5	39,199.0
0.2	9.58440e+008	8.00103e+008	9.56401e+008	8.22837e+008	45,970.3	38,388.7	26,706.6	45,858.4	26,667.8	39,480.0
0.4	9.61255e+008	8.03734e+008	9.60346e+008	8.26575e+008	46,132.7	38,587.7	26,894.2	46,093.5	26,882.9	39,684.4
0.6	9.62579e+008	8.05815e+008	9.62402e+008	8.28876e+008	46,197.2	38,675.2	26,999.3	46,187.2	26,999.4	39,789.7
0.8	9.62867e+008	8.06304e+008	9.62888e+008	8.29527e+008	46,218.7	38,698.6	27,035.8	46,216.5	27,040.3	39,815.4
1.0	9.62982e+008	8.06399e+008	9.62982e+008	8.29631e+008	46,223.2	38,707.1	27,046.0	46,223.2	27,046.0	39,822.3
0.8	9.63095e+008	8.06489e+008	9.63105e+008	8.29763e+008	46,229.9	38,712.5	27,059.0	46,229.8	27,056.5	39,831.7
0.6	9.63674e+008	8.06873e+008	9.63546e+008	8.30540e+008	46,271.2	38,738.6	27,104.0	46,257.6	27,097.6	39,871.9
0.4	9.66744e+008	8.08801e+008	9.65293e+008	8.32933e+008	46,388.5	38,824.9	27,230.0	46,341.5	27,203.4	39,985.4
0.2	9.71827e+008	8.12389e+008	9.68315e+008	8.37217e+008	46,638.3	39,034.6	27,444.2	46,508.0	27,385.4	40,183.7
0.0	9.75586e+008	8.15964e+008	9.71246e+008	8.40469e+008	46,988.3	39,318.6	27,764.1	46,741.3	27,657.2	40,469.7

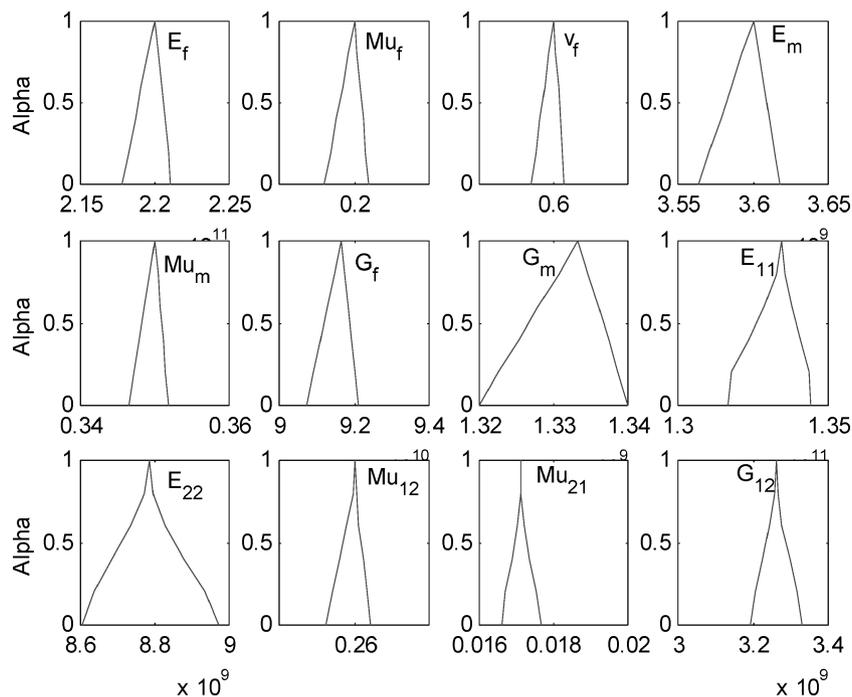


Fig. 8 Membership functions of the parameters of the beam.

Table 5 Nodal displacements of the cantilever beam (meters)

α	Node 1: $x = 0.2$ m	Node 2: $x = 0.4$ m	Node 3: $x = 0.6$ m	Node 4: $x = 0.8$ m	Node 5: $x = 1.0$ m	Node 6: $x = 1.2$ m	Node 7: $x = 1.4$ m	Node 8: $x = 1.6$ m	Node 9: $x = 1.8$ m	Node 10: $x = 2.0$ m
0.0	$2.69900e-006$	$1.02243e-005$	$2.14241e-005$	$3.54280e-005$	$5.16020e-005$	$6.92896e-005$	$8.76595e-005$	0.000107074	0.000127531	0.000145628
0.2	$2.69900e-006$	$1.02243e-005$	$2.14241e-005$	$3.54280e-005$	$5.16020e-005$	$6.92896e-005$	$8.76595e-005$	0.000107074	0.000127531	0.000145628
0.4	$2.69900e-006$	$1.02243e-005$	$2.14241e-005$	$3.54280e-005$	$5.16020e-005$	$6.92896e-005$	$8.76595e-005$	0.000107074	0.000127531	0.000145628
0.6	$2.69900e-006$	$1.02243e-005$	$2.14241e-005$	$3.54280e-005$	$5.16020e-005$	$6.92896e-005$	$8.76595e-005$	0.000107074	0.000127531	0.000145628
0.8	$2.79066e-006$	$1.05519e-005$	$2.21707e-005$	$3.67879e-005$	$5.36324e-005$	$7.19021e-005$	$9.10310e-005$	0.000110878	0.000131438	0.000150875
1.0	$2.85041e-006$	$1.07454e-005$	$2.25623e-005$	$3.74040e-005$	$5.44706e-005$	$7.30854e-005$	$9.26948e-005$	0.000112868	0.000133298	0.000153799
0.8	$2.89575e-006$	$1.09069e-005$	$2.29853e-005$	$3.80764e-005$	$5.54908e-005$	$7.43321e-005$	$9.42207e-005$	0.000114362	0.000135179	0.000155997
0.6	$2.99645e-006$	$1.12292e-005$	$2.36704e-005$	$3.90776e-005$	$5.69972e-005$	$7.64205e-005$	$9.75270e-005$	0.000118177	0.000139569	0.000161063
0.4	$2.99645e-006$	$1.12292e-005$	$2.36704e-005$	$3.90776e-005$	$5.69972e-005$	$7.64205e-005$	$9.75270e-005$	0.000118177	0.000139569	0.000161063
0.2	$2.99645e-006$	$1.12292e-005$	$2.36704e-005$	$3.90776e-005$	$5.69972e-005$	$7.64205e-005$	$9.75270e-005$	0.000118177	0.000139569	0.000161063
0.0	$2.99645e-006$	$1.12292e-005$	$2.36704e-005$	$3.90776e-005$	$5.69972e-005$	$7.64205e-005$	$9.75270e-005$	0.000118177	0.000139569	0.000161063

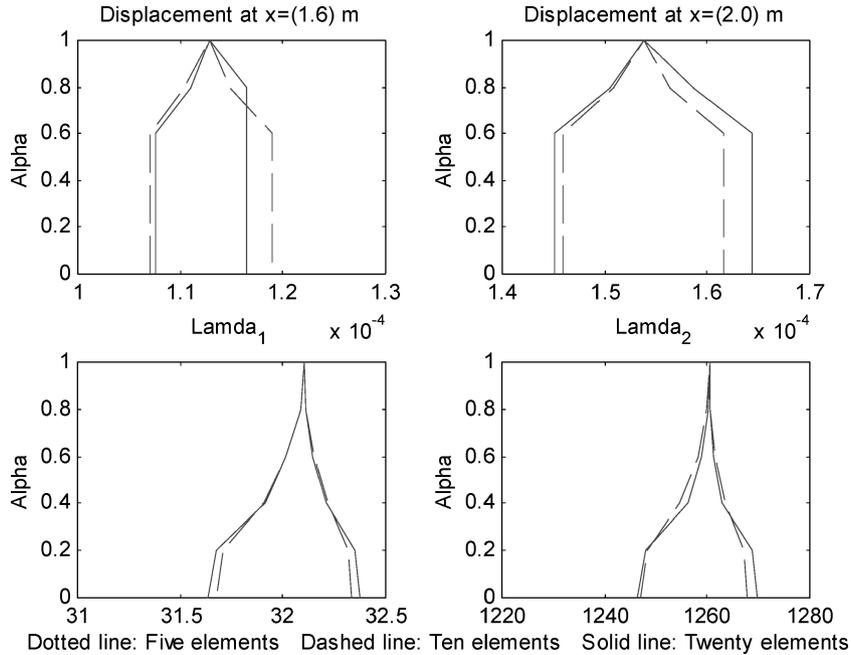


Fig. 9 Convergence of fuzzy FEM.

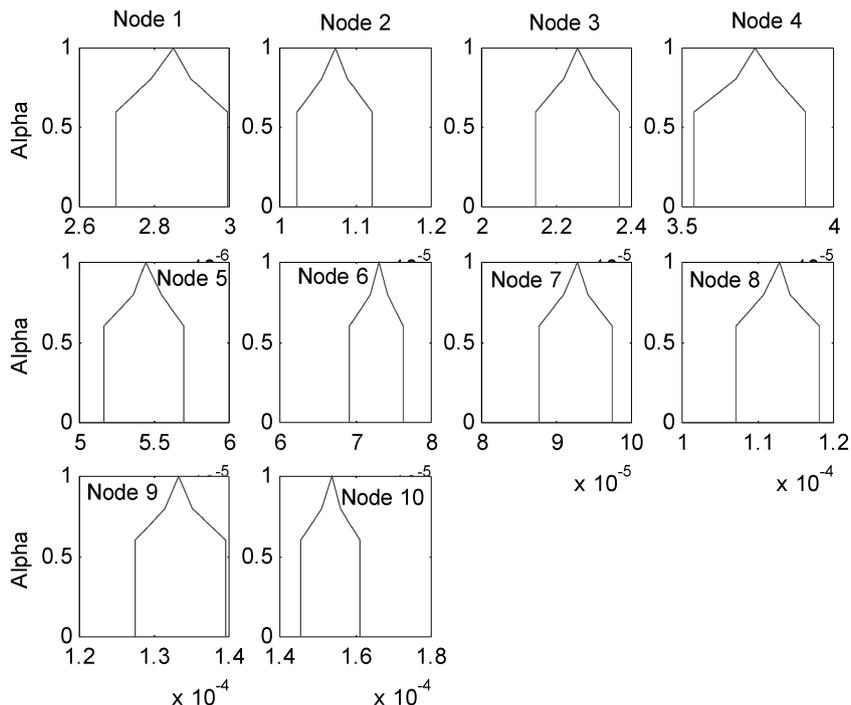


Fig. 10 Membership functions of the nodal displacements of the cantilever beam.

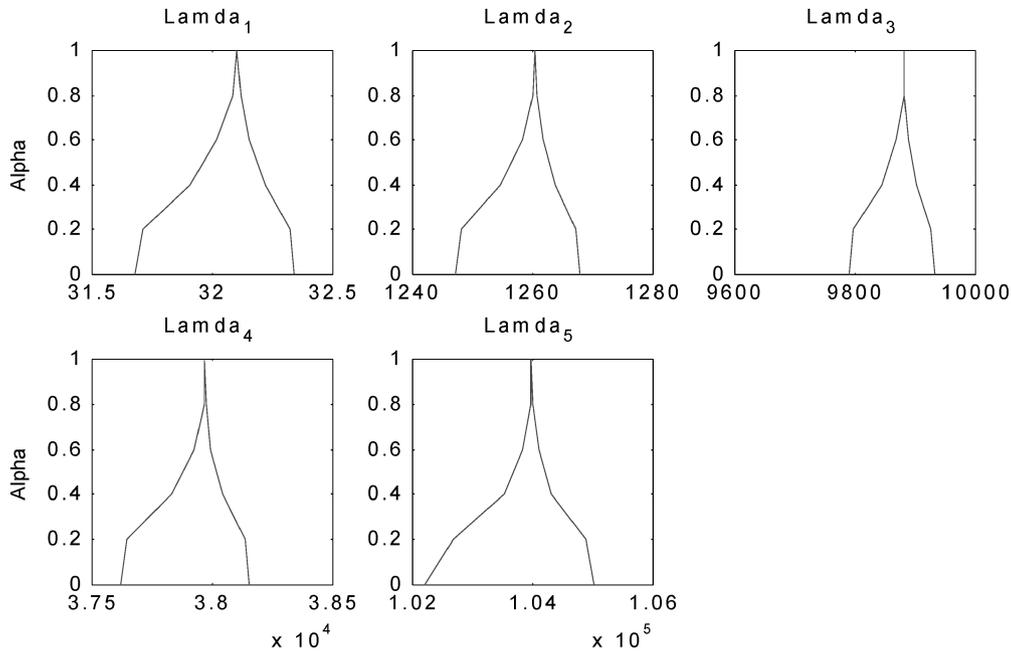


Fig. 11 Membership functions of the eigenvalues of the beam.

Table 6 Membership functions of the first five eigenvalues of the beam, $\lambda = \omega^2$ (rad/s)²

α	λ_1	λ_2	λ_3	λ_4	λ_5
0.0	32.3376	1,267.82	9,930.56	38,154.7	105,011
0.2	32.3205	1,267.21	9,926.08	38,137.7	104,887
0.4	32.2189	1,263.65	9,900.48	38,040.5	104,291
0.6	32.1549	1,261.61	9,888.09	37,993.4	104,086
0.8	32.1165	1,260.64	9,882.42	37,972.1	104,000
1.0	32.1030	1,260.47	9,881.38	37,968.4	103,983
0.8	32.0857	1,260.04	9,879.75	37,961.9	103,961
0.6	32.0162	1,258.20	9,869.24	37,920.5	103,816
0.4	31.9043	1,254.69	9,845.14	37,826.9	103,511
0.2	31.7124	1,248.25	9,798.27	37,647.8	102,673
0.0	31.6797	1,247.15	9,790.23	37,617.1	102,201

lamina of the beam contains 60 vol% of T-300 carbon fibers in an epoxy matrix. For the fiber, E_f is 220 GPa and ν_f is 0.2. For the matrix, E_m is 3.6 GPa and ν_m is 0.35. Consider the laminate as a [+45/-45]_s symmetric laminate with four layers. Each layer is 6 mm thick, the width of the beam is 24 mm, the length is 2 m so that the length of each element is 0.2 m, and the transverse distributed load is $q = 1$ N/m. The positive z direction is assumed to be pointing down in the direction of q .

For the finite element analysis, the membership functions of the fundamental parameters used are given in Fig. 8. A convergence study is conducted by changing the numbers of elements used in the model. Figure 9 shows a comparison of the results given by three fuzzy finite element models with 5, 10, and 20 elements. This graph shows the convergence of the results given by the fuzzy finite element method.

In the subsequent analysis, 10 finite elements are used. The stiffness matrices $[\bar{Q}]$ of the two layers are given in Tables 2 and 3, and matrices A , B , and D , are shown in Table 4. The computed beam displacements and eigenvalues are shown in Tables 5 and 6, respectively. Figures 10 and 11 give the graphical representation of the membership functions of beam displacements and eigenvalues implied by the results of Tables 5 and 6, respectively.

The fuzzy finite element results given in Tables 5 and 6 are compared with those obtained from a deterministic analysis of the classical laminated beam theory. The crisp values of the analytic results with length = 2.0 m, yield the tip deflection of the beam as 0.000153799 m, and the smallest eigenvalue as 32.0976 (rad/s)².

The finite element method with 10 elements gave the value of deflection as 0.000153799 m for $\alpha = 1$ with a fuzzy deviation for values of $0 \leq \alpha < 1$, and the value of the smallest eigenvalue as 32.103 (rad/s)² for $\alpha = 1$ with a fuzzy deviation for values of $0 \leq \alpha < 1$. It can be seen that the crisp value of the tip deflection of the beam is exactly same as the value given by the fuzzy finite element analysis with $\alpha = 1$ while the crisp value of the smallest eigenvalue is very close to the value given by the fuzzy finite element method with $\alpha = 1$.

VI. Conclusions

A fuzzy finite element analysis of laminated composite beams has been introduced in this work. From the analysis and computational results presented, the following observations can be made:

- 1) For triangular form of membership functions of input parameters, the membership functions of the computed results might not remain triangular; they may have a distorted form.
- 2) When the parameters/characteristics of the fuzzy composite material change within certain ranges, we can associate a suitable range or interval of confidence for each α presumption.
- 3) The computed results will involve much larger deviations than those of the inputs, where the deviation of a fuzzy number is defined as

$$\text{deviation} = \frac{(\text{value at } \alpha = 0^+) - (\text{value at } \alpha = 0^-)}{\text{nominal value of the fuzzy number}}$$

For an input deviation of about 1%, the deviation of the results has been observed to be as much as 5–10% or even larger in some cases. This is a reflection of the inherent characteristics of fuzzy computations. Each computational step with fuzzy numbers will result in a wider range than the previous step. Therefore, some compression and truncation methods must be considered to make the final result reasonable and more accurate when larger uncertainties are involved.

- 4) The method is shown to be converged. If we want to obtain a better result from the fuzzy finite element method, a finer mesh or a higher-order interpolation model can be applied.

It has been observed that the tip displacement and natural frequency results given by the fuzzy finite element method with $\alpha = 1$ are either the same or very close to the crisp values given by the classical laminate beam theory. When fuzziness is introduced in the structural modeling, the resulting finite element results are expected to be more realistic and reflect the true characteristics and behavior

of the composite beam or structure. The fuzzy finite element technique presented can be used to derive a state-space model that can be used in structural design and control. Although the analysis of composite beams is conducted using the Euler–Bernoulli beam theory in this work, the methodology is applicable to Rayleigh or Timoshenko beam theories²⁰ as well.

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